1. Give an example of 3 events A, B, C which are pairwise independent but not independent. Hint: find an example where whether C occurs is completely determined if we know whether A occurred and whether B occurred, but completely undetermined if we know only one of these things.

Answer :

Here's an example of three events AAA, BBB, and CCC that are pairwise independent but not mutually independent:

**Example:**

Let AAA, BBB, and CCC be defined as follows in the context of flipping two fair coins:

* **Event AAA**: The first coin is heads.
* **Event BBB**: The second coin is heads.
* **Event CCC**: Either both coins are heads or both coins are tails.

**Probability Calculation:**

* **Pairwise Independence**:
  + AAA and BBB are independent because the outcome of one coin does not affect the outcome of the other. P(A∩B)=14P(A \cap B) = \frac{1}{4}P(A∩B)=41​, and P(A)×P(B)=12×12=14P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}P(A)×P(B)=21​×21​=41​.
  + AAA and CCC are independent:
    - P(A)=12P(A) = \frac{1}{2}P(A)=21​ (probability that the first coin is heads).
    - P(C)=12P(C) = \frac{1}{2}P(C)=21​ (probability that both coins are the same).
    - P(A∩C)=14P(A \cap C) = \frac{1}{4}P(A∩C)=41​ (probability that the first coin is heads and both coins are the same, meaning the second coin is also heads).
    - P(A)×P(C)=12×12=14P(A) \times P(C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}P(A)×P(C)=21​×21​=41​.
  + BBB and CCC are independent by similar reasoning:
    - P(B)=12P(B) = \frac{1}{2}P(B)=21​ (probability that the second coin is heads).
    - P(B∩C)=14P(B \cap C) = \frac{1}{4}P(B∩C)=41​ (probability that the second coin is heads and both coins are the same, meaning the first coin is also heads).
    - P(B)×P(C)=12×12=14P(B) \times P(C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}P(B)×P(C)=21​×21​=41​.
* **Not Mutually Independent**:
  + For mutual independence, we would need P(A∩B∩C)=P(A)×P(B)×P(C)P(A \cap B \cap C) = P(A) \times P(B) \times P(C)P(A∩B∩C)=P(A)×P(B)×P(C).
  + P(A∩B∩C)=P(first coin is heads and second coin is heads)=14P(A \cap B \cap C) = P(\text{first coin is heads and second coin is heads}) = \frac{1}{4}P(A∩B∩C)=P(first coin is heads and second coin is heads)=41​.
  + However, P(A)×P(B)×P(C)=12×12×12=18P(A) \times P(B) \times P(C) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}P(A)×P(B)×P(C)=21​×21​×21​=81​.

Since 14≠18\frac{1}{4} \neq \frac{1}{8}41​=81​, the events AAA, BBB, and CCC are not mutually independent, but they are pairwise independent.

This example demonstrates that CCC is completely determined if both AAA and BBB are known, but is undetermined if only one of AAA or BBB is known.

1. A bag contains one marble which is either green or blue, with equal probabilities. A green marble is put in the bag (so there are 2 marbles now), and then a random marble is taken out. The marble taken out is green. What is the probability that the remaining marble is also green?

**Step 1: Define Events**

* **Let G1G\_1G1​** be the event that the initial marble was green.
* **Let G2G\_2G2​** be the event that the initial marble was blue (so the added marble is green).
* **Let GGG** be the event that a green marble is drawn.

We want to find P(G1∣G)P(G\_1 | G)P(G1​∣G), the probability that the remaining marble is green given that a green marble was drawn.

**Step 2: Calculate Prior Probabilities**

* P(G1)=12P(G\_1) = \frac{1}{2}P(G1​)=21​: Probability that the initial marble is green.
* P(G2)=12P(G\_2) = \frac{1}{2}P(G2​)=21​: Probability that the initial marble is blue.

**Step 3: Likelihood of Drawing a Green Marble**

* **If the initial marble was green (G1)(G\_1)(G1​)**, there are two green marbles in the bag, so the probability of drawing a green marble is: P(G∣G1)=1P(G | G\_1) = 1P(G∣G1​)=1
* **If the initial marble was blue (G2)(G\_2)(G2​)**, there is one green and one blue marble, so the probability of drawing a green marble is: P(G∣G2)=12P(G | G\_2) = \frac{1}{2}P(G∣G2​)=21​

**Step 4: Apply Bayes' Theorem**

Bayes' Theorem states:

P(G1∣G)=P(G∣G1)×P(G1)P(G)P(G\_1 | G) = \frac{P(G | G\_1) \times P(G\_1)}{P(G)}P(G1​∣G)=P(G)P(G∣G1​)×P(G1​)​

Where P(G)P(G)P(G) is the total probability of drawing a green marble:

P(G)=P(G∣G1)×P(G1)+P(G∣G2)×P(G2)P(G) = P(G | G\_1) \times P(G\_1) + P(G | G\_2) \times P(G\_2)P(G)=P(G∣G1​)×P(G1​)+P(G∣G2​)×P(G2​) P(G)=1×12+12×12=12+14=34P(G) = 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}P(G)=1×21​+21​×21​=21​+41​=43​

Now, compute P(G1∣G)P(G\_1 | G)P(G1​∣G):

P(G1∣G)=1×1234=1234=23P(G\_1 | G) = \frac{1 \times \frac{1}{2}}{\frac{3}{4}} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}P(G1​∣G)=43​1×21​​=43​21​​=32​

**Final Answer:**

The probability that the remaining marble is also green is **23\frac{2}{3}32​**.